Spin Models
and Superconformal Yang-Mills Theory

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We apply novel techniques in planar superconformal Yang-Mills theory which stress the role of the Yangian algebra. We compute the first two Casimirs of the Yangian, which are identified with the first two local abelian Hamiltonians with periodic boundary conditions, and show that they annihilate the chiral primary states. We streamline the derivation of the R-matrix in a conventional spin model, and extend this computation to the gauge theory. We comment on higher-loop corrections and higher-loop integrability.

1. Introduction

Planar superconformal gauge theory in four dimensions has yielded to a partial description in terms of integrable spin model structures. In the $N \to \infty$ limit, the $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory has a dilatation operator whose contribution at one-loop in $g^2 N$ can be identified with the Hamiltonian of an integrable spin chain [1-24]. Furthermore, the ordinary $PSU(2, 2|4)$ symmetry generators of the gauge theory at tree level, $g^2 N = 0$, can be identified with the total spin variables $J^A = \sum_{i=1} J_i^A$, where i labels the sites of the chain or lattice, and A runs over the dimension of the symmetry group. Spin models were further exploited in [25,26] to uncover additional tree level operators in the gauge theory, $Q^A = f_{BC}^A \sum_{j<k} J_j^B J_k^C$, which generate the higher non-local charges $\mathcal{J}_n^A$, $n = 2, 3, ...$, of a Yangian algebra [27-35]. Evidence for this larger Yangian symmetry had already been seen, at least at $g^2 N = \infty$, in the AdS/CFT dual theory described by the classical Green-Schwarz superstring action for $AdS_5 \times S^5$ [36-38]. Another element of integrability in the gauge theory is the R-matrix, which was extrapolated from a subsector [3]. There are also one-loop higher local commuting Hamiltonians which were computed in some special subsectors of the gauge theory [5].

With these identifications, it is compelling to ask how much more of the gauge theory can be cast in integrable forms that could eventually reveal the exact spectrum and correlation functions.

In this paper, we apply novel techniques in planar superconformal Yang Mills theory to analyze the integrable structure, and also show how they streamline the calculations of conventional spin models [39-40]. We focus on the role of the Yangian in the gauge theory, and explain why it is useful even though it cannot be defined for periodic spin chains. We use it to compute the R-matrix $R_{ij}(u)$, and to define the local commuting Hamiltonians H_κ with periodic boundary conditions, by identifying them as Casimirs of the Yangian.

We compute the second Casimir explicitly in the full gauge theory, and show that it annihilates the chiral primary states. We conjecture that *all* the Casimirs of the Yangian annihilate the chiral primaries. Since the supergravity states in the AdS/CFT correspondence are chiral primary states, this explains why one does not see the Yangian symmetry in the supergravity Lagrangian.

As an introduction to our methods, we first consider cases of more conventional quantum spin models and show how the Yangian can be used in those models to find the R-matrix and Hamiltonians. In sections 2 and 3, we treat the $SU(2)$ XXX-model in the spin 1/2 representation, and the $SL(2)$ chain in the spin s representation.

In section 4 we compute both the R-matrix and the Hamiltonian directly from the Yangian for the $PSU(2,2|4)$ spin chain where the single-site spin variables are in the representation given by the one-particle states in free $\mathcal{N} = 4$ super Yang-Mills theory.

The second Casimir of the Yangian is computed in section 5. In conventional integrable models, the abelian Hamiltonians are found via the monodromy matrix, a useful holonomy that is constructed from the R-matrix in terms of the local dynamical variables J_i^A. In classical integrable models, expansion of the monodromy matrix around $u = \infty$ in inverse powers of the spectral parameter gives the Yangian generators and some non-local abelian charges. The monodromy matrix has an isolated singularity at $u = 0$, which may be separated from the regular part of the expression by rewriting the matrix as a product of three local factors [31,41]. The trace of the monodromy can be shown then to have just one local factor, when periodic boundary conditions are applied. In this way, the reexpansion of the trace of the monodromy around $u = 0$ gives the commuting local conserved quantities for the periodic chain.

This procedure is somewhat modified in quantum integrable models, due to the ambiguity in ordering the quantum operators. It is convenient to multiply the R-matrix by u. Again there are two sets of abelian conserved charges, one is local and one is non-local. Expanding the monodromy matrix in powers of the spectral parameter gives the non-local Yangian generators and non-local abelian charges [32], where the latter can also be derived by simply expanding the trace of the monodromy matrix, the transfer matrix. Expanding the logarithim of the transfer matrix gives the local commuting Hamiltonians [42-44].

But in the planar Yang-Mills theory, it is difficult to perform these expansions and recover explicit expressions for the higher local abelian conserved charges. We are thus led to present an alternative derivation by finding them as Casimirs of the Yangian.

In section 6, the second Casimir is shown to annihilate the chiral primary states. This is consistent with our experience that all one-loop corrections are related to the one-loop anomalous dimension, which vanishes for chiral primary states.

In section 7, we discuss how extending the Yangian to higher orders in g^2N addresses higher-loop integrability. We have conjectured in [25] that the Yangian generators can be defined to all orders in g^2N via $\tilde{J}^A = J^A + g^2N J_2^A + \ldots$, $\tilde{Q}^A = Q^A + g^2N Q_2^A + \ldots$, etc., although $\tilde{Q}^A$ will not have the simple bilinear form that it has at tree level. The planar $\mathcal{N} = 4$ Yang-Mills theory (SYM) is radially quantized on $R \times S^3$, with the Hamiltonian given by the dilatation operator $\tilde{D}$, one of the ordinary symmetries of $PSU(2,2|4)$. The first spin chain Hamiltonian H_1 is interpreted as the one-loop g^2N contribution to the dilatation

operator, D_2. The spin chain higher Hamiltonians H_κ, $\kappa > 1$, which we introduce in section 6 as Casimirs, are interpreted as one-loop contributions to other operators in SYM, which have zero tree-level contribution. They have been used to provide an understanding of certain degeneracies of the one-loop anomalous dimension for charge-conjugated states [5].

Their higher-loop contributions will be Casimirs of the higher-loop Yangian charges, but we do not calculate any higher-loop corrections here. Since the R-matrix and monodromy satisfy algebraic constraints and are derived from a universal R-matrix which is an element of the Yangian algebra, the algebra is important for studying any choice of boundary conditions.

Appendix A adapts some arguments developed for the $PSU(2,2|4)$ gauge theory to the spin s $SU(2)$ chain.

2. $XXX_{1/2}$ Model

We recall the role of Yangian symmetry in integrable models by first considering the familiar quantum $XXX_{1/2}$ model, the ordinary Heisenberg spin chain, to review and fix the nomenclature. For L sites, its operators are defined in a Hilbert space $\mathcal{H}_L = \bigotimes_{i=1}^{L} h_i$. At each site i, the local spin variable J_i^A is in the spin $\frac{1}{2}$ representation of $SU(2)$ and acts as $\frac{\sigma^A}{2i}$ on a space h_i that is spanned by two states, corresponding to up or down spin. σ^A are the Pauli matrices. We have

$$[J_i^A, J_j^B] = \epsilon_{ABC} J_j^C \delta_{ij} \,. \tag{2.1}$$

The spin chain Hamiltonian is

$$H = \beta \sum_{i=1}^{L} (J_i^A J_{i+1}^A + \frac{1}{4}) \,, \tag{2.2}$$

where positive (negative) β corresponds to the ferromagnetic (antiferromagnetic) case. For the moment, we do not impose periodic boundary conditions. This makes it possible to construct the Yangian generators $\mathcal{J}_n^A$, with $n = 0, \ldots, L$. The first two generators will be denoted as $\mathcal{J}_0^A = J^A, \mathcal{J}_1^A = Q^A$, and $\mathcal{J}_n^A$ is an operator that acts on $n+1$ sites at a time and arises from commutators of the Q^A. The total spin variables

$$J^A = \sum_{i=1}^{L+1} J_i^A \tag{2.3}$$

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are the ordinary $SU(2)$ symmetry generators, with

$$[J^A, J^B] = \epsilon_{ABC} J^C . \tag{2.4}$$

The charges J^A commute with the Hamiltonian

$$[H, J^A] = 0 , \tag{2.5}$$

due to the $SU(2)$ symmetry of the model. The bilocal Yangian generators are represented by

$$Q^A = \epsilon_{ABC} \sum_{1 \le i < j \le L+1} J_i^B J_j^C . \tag{2.6}$$

They act on two sites at a time like the Hamiltonian, but involve pairs that are not necessarily nearest neighbors, as well as the group structure constants. Their commutation with the ordinary symmetry generators is

$$[J^A, Q^B] = \epsilon_{ABC} Q^C . \tag{2.7}$$

We recall that their commutation with the Hamiltonian is given by

$$[H, Q^A] = \frac{1}{2} \beta \left(J_1^A - J_{L+1}^A \right) , \tag{2.8}$$

which can be shown by first considering a system of two spins, and proving

$$[H_{12}, Q_{12}^A] = \frac{1}{2} \beta \, q_{12}^A , \tag{2.9}$$

where the two-body operators are $H_{i\,i+1} = \beta(J_i^A J_{i+1}^A + \frac{1}{4})$, $Q_{ij}^A = \epsilon_{ABC} J_i^B J_j^C$, and q_{ij}^A is the difference operator

$$q_{ij}^A = J_i^A - J_j^A . \tag{2.10}$$

We remark that (2.9) follows simply from the properties of the spin $\frac{1}{2}$ representation at each site, since for $J_j^A = \frac{\sigma_j^A}{2i}$ we have

$$\begin{aligned}
[H_{12}, Q_{12}^A] &= \beta \, \epsilon_{ABC} [J_1^D J_2^D, J_1^B J_2^C] \\
&= \frac{1}{2} \beta \left(J_1^A - J_2^A \right) .
\end{aligned} \tag{2.11}$$

For a chain of $L + 1$ spins, the commutator $[H, Q^A]$ is given by

$$[H, Q^A] = \sum_{i=1}^{L} [H_{i,i+1}, Q_{i,i+1}^A] = \frac{\beta}{2} \sum_{i=1}^{L} q_{i,i+1}^A = \frac{1}{2} \beta \, q^A , \tag{2.12}$$

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where $q^A = J_1^A - J_{L+1}^A$, since the cross terms *cf.*[25] vanish,

$$0 = \left[H_{i,i+1} , \sum_{j<k,(j,k)\neq(i,i+1)} Q_{jk}^A \right] . \qquad (2.13)$$

This verifies the claim (2.8), that for finite chains, the Yangian commutes with the Hamiltonian H up to edge effects. For chains of infinite length where we ignore the lattice total derivative, the Yangian is an exact symmetry.

For a finite chain with periodic boundary conditions

$$J_i^A = J_{i+L}^A , \qquad (2.14)$$

we also have $[H, Q^A] = 0$. We will see in section 6 that this implies the Casimir operators of the Yangian can still be defined for the periodic chain, even though the Yangian representation (2.6) cannot. These Casimirs are local and their commutator with H vanishes. We note that (2.8) is the SU(2) spin $\frac{1}{2}$ analogue of the $[H, Q^A]$ commutator which was derived for the $PSU(2,2|4)$ chain in [25]. There an argument involving the properties of the two-particle modules for the planar gauge theory was needed, whereas here Pauli matrix identities suffice. In fact, as shown in [1], the Hamiltonian (2.2) can be used to find the one-loop anomalous dimensions for a subsector of the gauge theory, with the ferromagnetic choice $\beta = \frac{g^2 N}{4\pi^2}$ and the boundary conditions (2.14).

H is an integrable Hamiltonian. For periodic boundary conditions, this means that it belongs to a set of L commuting operators. In the thermodynamic limit $(L \to \infty)$, the abelian symmetry of the family of integrable Hamiltonians and the non-abelian Yangian symmetry become infinite-dimensional. The abelian Hamiltonians are given by the transfer matrix, the trace of the monodromy matrix, which also depends on L sites, and in turn is constructed from the R-matrix which depends on two sites. The R-matrix satisfies a Yang-Baxter equation. This implies certain relations between the elements of the monodromy matrix which can lead ultimately to the spectrum and correlation functions of the model, via the algebraic Bethe ansatz [40]. We review some of its features with a view toward introducing our Yangian methods.

The R-matrix is defined on two sites and is a function of the spectral parameter u

$$R_{jm}(u) = (u + \frac{i}{2})I_j \otimes I_m - J_j^A \otimes \sigma_m^A , \qquad (2.15)$$

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where J_i^A is the local spin variable introduced above, and σ_m^A are Pauli matrices defined at site m. Our convention is $R_{jm}(0) = iP_{jm}$, where P_{jm} permutes the sites at j and m:

$$P_{j,m} \equiv \tfrac{1}{2}(I_j \otimes I_m + \sigma_j^A \otimes \sigma_m^A) \,. \tag{2.16}$$

The non-abelian properties of the R-matrix (2.15) are given by the Yang-Baxter equation,

$$R_{mn}(u-v)R_{im}(u)R_{in}(v) = R_{in}(v)R_{im}(u)R_{mn}(u-v) \tag{2.17}$$

which is satisfied due to (2.1) and the familiar properties of the Pauli matrices, and encodes the Yangian symmetry. The sites i, j are any points on the quantum chain; and m, n label an auxiliary space. The R-matrix on two auxiliary sites has a similar form with $J_m^A \equiv \frac{\sigma_m^A}{2i}$. (2.15) is Yang's solution to the Yang-Baxter equation, and the one relevant for the $XXX_{\frac{1}{2}}$ model. The monodromy matrix $T_m(u)$ defines the transport from site 1 to site L, and for convenience is defined in terms of a Lax operator

$$L_{jm}(u) = R_{jm}(u - \frac{i}{2}) \,, \tag{2.18}$$

as

$$T_m(u) \equiv L_{L,m}(u) \ldots L_{1,m}(u) \,. \tag{2.19}$$

The shift in u makes the Bethe Ansatz equations more symmetric, but is not necessary to define monodromy. For periodic boundary conditions, its trace, the transfer matrix, produces commuting operators M_κ

$$F(u) = \operatorname{tr} T_m(u) = 2u^L + \sum_{\kappa=0}^{L-2} M_\kappa u^\kappa \,, \tag{2.20}$$

where the trace is on the matrix at site m. The commuting operators defined by M_κ are in general non-local. To access the *local* abelian Hamiltonians, one must take derivatives of the logarithm of the transfer matrix,

$$H_\kappa \sim \frac{d^\kappa}{du^\kappa} \ln F(u)|_{u=\frac{i}{2}} \tag{2.21}$$

in the fashion of keeping only connected graphs for Green functions in a field theory[42]. The $\sim$ defines H_κ up to overall multiplicative and additive constants. In particular, the first two local charges can be defined as H_1 given by (2.2), corresponding to $H_1 = \beta(-\frac{i}{2}\frac{d}{du}\ln F(u) + \frac{1}{2}L))|_{u=0}$, and

$$H_2 = \sum_{i=1}^{L}[P_{i,i+1}, P_{i+1,i+2}] \,. \tag{2.22}$$

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The monodromy also satisfies a Yang-Baxter relation

$$R_{mn}(u-v)T_m(u)T_n(v) = T_n(v)T_m(u)R_{mn}(u-v)\,, \tag{2.23}$$

which implies that $F(u)F(v) = F(v)F(u)$, so that $[H_\kappa, H_{\kappa'}] = 0$.

The monodromy matrix is a polynomial in u, and its components are the Yangian generators together with some non-local abelian charges, all of which can still be defined since $T_m(u)$ involves only L sites. For $eg.$, for $L = 2$

$$\begin{aligned}
T_m(u) &= L_{2m}(u)L_{1m}(u) \\
&= u^2 I_m - u(J_1^A + J_2^A)\sigma_m^A + J_2^B J_1^C(\sigma^B \sigma^C)_m \\
&= u^2 I_m - uJ^A \sigma_m^A - iQ^A \sigma_m^A + (\tfrac{1}{2}J^A J^A + \tfrac{3}{4})I_m\,.
\end{aligned} \tag{2.24}$$

With a view toward extending the monodromy to higher-loops in $g^2 N$, which we discuss in section 7, one might inquire how direct is the relation between the Yang-Baxter equation (2.23) and the Yangian. We can show for $T_m(u)$ given by (2.24), that (2.23) follows immediately from the Yangian commutation relations (2.4) and (2.7) and a two-site subsidiary condition

$$[Q^A, Q^B] = \epsilon_{ABC}(\tfrac{1}{2}J^D J^D J^C + \tfrac{3}{4}J^C)\,. \tag{2.25}$$

This extra condition (2.25) holds by inspection when J^A and Q^A are in the representation given by (2.3) and (2.6) with $J_j^A = \frac{\sigma_j^A}{2i}$. Given that the fundamental relation (2.23) holds for $L = 2$, it follows for arbitrary L due to the commutativity of R-matrices with no common sites, see $eg.$[40].

Note that for $L > 2$, the monodromy matrix will contain higher Yangian charges $\mathcal{J}_n^A$, for $n = 0, \ldots L-1$, where $\mathcal{J}_0^A = J^A, \mathcal{J}_1^A = Q^A, \ldots$. For general L, the monodromy can be expressed in terms of the Yangian generators by

$$T_{ab}(u) = u^L \delta_{ab} + \sum_{n=0}^{L} u^{L-n} t_{ab}^{(n)}\,, \tag{2.26}$$

where a, b label the indices of the matrix on the auxiliary site m and

$$\begin{aligned}
t_{ab}^{(0)} &= J^A T_{ab}^A \\
t_{ab}^{(1)} &= Q^A T_{ab}^A + \tfrac{1}{2}t_{ad}^{(0)}t_{db}^{(0)} + \alpha\delta_{ab} + \beta t_{ab}^{(0)}\,,
\end{aligned} \tag{2.27}$$

for some coefficients α, β. The coefficients $t_{ab}^{(n)}$ for $n > 1$ will involve the higher Yangian generators $\mathcal{J}_n^A$. It can be shown for all $N \geq 2$, when J_i^A (and T_{ab}^A) are in the N-dimensional

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representation of $SU(N)$, that the components of the monodromy satisfy commutation relations for $[t_{ab}^{(n)}, t_{cd}^{(m)}]$ which are equivalent to the Yangian defining relations [30]. The fact that the transfer matrix (2.26) satisfies the Yang-Baxter relation (2.23) for some suitable $SU(N)$ R-matrix again depends on the fact that J_i^A is in a special representation. So the extension of (2.26) and (2.27) to higher-loops in the gauge theory, where the representation will change, is not straightfoward, as we observe in section 7. Nonetheless at the one-loop level, knowledge of the monodromy matrix, whose logarithmic expansion gives the one-loop local Hamiltonians, is equivalent to the tree-level Yangian.

3. XXX_s Model

The purpose of this section is to derive the R-matrix and the Hamiltonian for the XXX_s model with methods similar to those used in [25]. This will streamline the computations and also allow us to generalize the results to $PSU(2,2|4)$. The XXX_s model is a spin chain where the local spin variables J_i^A take values in the $2s+1$-dimensional space $\mathcal{C}^{2s+1}$, and $s = 0, \frac{1}{2}, 1, \ldots$ is the spin labeling representations of $SU(2)$. Following the literature [39,40], we look for an R-matrix that leads to monodromy with integrable local Hamiltonians. The equation for the R-matrix follows from the the Yang-Baxter relation for a universal R-matrix $\mathcal{R}$ and acts in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, when $\mathcal{A}$ is the Yangian algebra of $SU(2)$:

$$\mathcal{R}_{12}\mathcal{R}_{32}\mathcal{R}_{31} = \mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}_{12}. \tag{3.1}$$

Since $\mathcal{R}$ is an element in $\mathcal{A} \otimes \mathcal{A}$, its representation has the form

$$R_{s's}(u-v) = (\rho(s,u) \otimes \rho(s',v))\,\mathcal{R}_{12}, \tag{3.2}$$

where in general s, s' label the spin of the $SU(2)$ representations at two sites [40]. For spin s at site i and spin $\frac{1}{2}$ at site m, the representation acts as

$$(I \otimes \rho(s,u) \otimes \rho(\tfrac{1}{2},v))\,\mathcal{R}_{32} = R_{im}(v-u) = (v-u+\frac{i}{2})I_i \otimes I_m - J_i^A \otimes \sigma_m^A. \tag{3.3}$$

Using standard procedures [40], one applies the representation $\rho(s,\lambda) \otimes \rho(s,\mu) \otimes \rho(\tfrac{1}{2},\sigma)$ to (3.1) to find a linear equation for $R_{ij}(\lambda)$:

$$R_{ij}(\lambda-\mu)\left((\sigma-\mu+\frac{i}{2})I_i \otimes I_m - J_i^A \otimes \sigma_m^A\right)\left((\sigma-\lambda+\frac{i}{2})I_j \otimes I_m - J_j^B \otimes \sigma_m^B\right)$$
$$= \left((\sigma-\lambda+\frac{i}{2})I_j \otimes I_m - J_j^B \otimes \sigma_m^B\right)\left((\sigma-\mu+\frac{i}{2})I_i \otimes I_m - J_i^A \otimes \sigma_m^A\right)R_{ij}(\lambda-\mu). \tag{3.4}$$

Here the spin variables on the quantum space J_i^A, J_j^B are taking values in $\mathcal{C}^{2s+1}$. A general representation $\rho(s_1, \lambda) \otimes \rho(s_2, \mu) \otimes \rho(s_3, \sigma)$ acting on (3.1) would mean that the single-site spin variables at the first, second, and third sites are in the representations s_1, s_2, s_3 respectively. (Note the permutation of indices between (2.17) and (3.4).) Now we diverge from the standard derivation. Labeling the sites i, j as $1, 2$, and letting $\mu = 0$, we rewrite (3.4) in terms of the two-site Yangian generators as

$$
\begin{aligned}
&R_{12}(\lambda)((iQ^A + \lambda J_1^A - (\sigma + \frac{i}{2})J^A)\sigma_m^A + J_1^A J_2^A I_m) \\
&= ((-iQ^A + \lambda J_1^A - (\sigma + \frac{i}{2})J^A)\sigma_m^A + J_1^A J_2^A I_m)R_{12}(\lambda),
\end{aligned}
\tag{3.5}
$$

where in this section we write $J^A = J_1^A + J_2^A$, $Q^A = \epsilon_{ABC} J_1^A J_2^B$. We will look for a solution where the R-matrix depends on J_1^A, J_2^A only through the Casimir, $i.e.$ $R_{12}(\lambda, J_1^A J_2^A)$. For spin s, the Casimir of the $SU(2)$ symmetry acting on a two-particle state is

$$
J^A J^A = -2s(s+1)I_1 \otimes I_2 + 2J_1^A J_2^A = -J_{12}(J_{12}+1) \equiv -J(J+1).
\tag{3.6}
$$

Let V_S be the $2s+1$-dimensional space of one-particle states (the states at one spin site). The tensor product $V_S \otimes V_S$ decomposes into irreducible representations of $SU(2)$,

$$
V_S \otimes V_S = \bigoplus_{j=0}^{2s} V_j.
\tag{3.7}
$$

The operator J in (3.6) has eigenvalue j acting on a two-particle state $|\kappa(j)\rangle$ that is contained in V_j, and $j = 0, 1, \ldots 2s$. Since $[J^A, J_1^D J_2^D] = 0$, and $[R_{12}, J_1^D J_2^D] = 0$, (3.5) becomes

$$
R_{12}(\lambda)(iQ^A + \lambda J_1^A) = (-iQ^A + \lambda J_1^A)R_{12}(\lambda),
\tag{3.8}
$$

or equivalently, since $[R_{12}, J^A] = 0$,

$$
R_{12}(\lambda)(2iQ^A + \lambda(J_1^A - J_2^A)) = (-2iQ^A + \lambda(J_1^A - J_2^A))R_{12}(\lambda).
\tag{3.9}
$$

To solve (3.8), define a permutation operator P_{ij} in $\mathcal{C}^{2s+1} \otimes \mathcal{C}^{2s+1}$ such that

$$
P_{12}J_1^A P_{12} = J_2^A, \qquad P_{12}J_2^A P_{12} = J_1^A, \qquad P_{12}P_{12} = 1.
\tag{3.10}
$$

(For $s = \frac{1}{2}$, the permutation P_{12} is given by (2.16)). We let $R_{12}(\lambda) = r(\lambda, J)P_{12}$, and find that (3.8) reduces to

$$
r(\lambda, J)(2iQ^A + \lambda q^A) = (2iQ^A - \lambda q^A)r(\lambda, J),
\tag{3.11}
$$

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with

$$q^A = J_1^A - J_2^A \,, \tag{3.12}$$

since $P_{12}Q^A P_{12} = -Q^A$ and $P_{12}q^A P_{12} = -q^A$. Using the identity

$$[J^D J^D, q^A] = 4Q^A \,, \tag{3.13}$$

we can evaluate (3.11) acting on a two-particle state $|\kappa(j)\rangle$ that is contained in V_j (and so has eigenvalues of $J^D J^D$ given above). Then

$$r(\lambda, J)(\frac{i}{2}J^D J^D q^A - \frac{i}{2}q^A J^D J^D + \lambda q^A)|\kappa(j)\rangle = (\frac{i}{2}J^D J^D q^A - \frac{i}{2}q^A J^D J^D - \lambda q^A)r(\lambda, J)|\kappa(j)\rangle \,. \tag{3.14}$$

From [25], we know that the action of q^A on a state in V_j can be written as a linear combination of states in V_{j-1} and V_{j+1}, i.e. for any $|\kappa(j)\rangle \in V_j$ we have

$$q^A|\kappa(j)\rangle = |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle \,, \tag{3.15}$$

where $|\chi^A(j-1)\rangle \in V_{j-1}$ and $|\rho^A(j+1)\rangle \in V_{j+1}$. Actually in [25] we proved (3.15) for the two-particle modules of the $PSU(2,2|4)$ gauge theory. But on inspection (3.15) also holds for an arbitrary spin s representation of $SU(2)$ due to the $SU(2)$ tensor product decomposition. See Appendix A. Using (3.6), we have $J^D J^D|\kappa(j)\rangle = -j(j+1)|\kappa(j)\rangle$, and computing (3.14) we derive

$$r(\lambda, J)(\frac{i}{2}J^D J^D q^A - \frac{i}{2}q^A J^D J^D + \lambda q^A)|\kappa(j)\rangle$$
$$= r(\lambda, j-1)(\lambda + ij)|\chi^A(j-1)\rangle + r(\lambda, j+1)(\lambda - i(j+1))|\rho^A(j+1)\rangle \,, \tag{3.16}$$

and

$$(\frac{i}{2}J^D J^D q^A - \frac{i}{2}q^A J^D J^D - \lambda q^A)r(\lambda, J)|\kappa(j)\rangle$$
$$= (-\lambda + ij)r(\lambda, j)|\chi^A(j-1)\rangle + (-\lambda - i(j+1))r(\lambda, j)|\rho^A(j+1)\rangle \,. \tag{3.17}$$

Equating the coefficients of $|\chi^A(j-1)\rangle$, we find

$$(\lambda + ij)r(\lambda, j-1) = (-\lambda + ij)r(\lambda, j) \,, \tag{3.18}$$

and an equivalent equation for the coefficients of $|\rho^A(j+1)\rangle$. The solution to (3.18) is

$$r(\lambda, j) = \frac{\Gamma(j+1-i\lambda)}{\Gamma(j+1+i\lambda)} \,, \tag{3.19}$$

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so the R-matrix that satisfies (3.5) is given by

$$R_{12}(\lambda) = \frac{\Gamma(J+1-i\lambda)}{\Gamma(J+1+i\lambda)} \, P_{12} \,. \tag{3.20}$$

This is the standard expression for the R-matrix whose monodromy generates the local abelian charges for the XXX_s model with periodic boundary conditions, but we have derived it here by considering how the Yangian acts on it.

The integrable Hamiltonian can now be found [40,39] from the spectral invariants of the monodromy constructed from the R-matrix in (3.20),

$$T_f = R_{L\,f}(\lambda)R_{L-1\,f}(\lambda)\dots R_{1f}(\lambda)\,, \tag{3.21}$$

i.e. from the transfer matrix $F_f(\lambda) = \mathrm{tr}_f T_f(\lambda)$ which satisfies $[F_f(\lambda), F_f(\mu)] = 0$. Hence

$$H \sim -i\frac{d}{d\lambda}\ln F_f(\lambda)|_{\lambda=0} = -\sum_{i=1}^{L} 2\psi(J_{i,i+1}+1)\,, \tag{3.22}$$

since $i\frac{d}{d\lambda}\ln r(J_{j,j+1}, \lambda)|_{\lambda=0} = 2\psi(J_{j,j+1}+1)$, where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$.

Alternatively, we will show that we can derive the Hamiltonian *directly* from the Yangian, without going through the R-matrix, by requiring that H_{12} be some function of the Casimir J that satisfies

$$[H_{12}(J), Q^A] = q^A \,. \tag{3.23}$$

Evaluating (3.23) via (3.13), we have

$$
\begin{aligned}
&[H_{12}(J), Q^A]|\kappa(j)\rangle \\
&= \frac{1}{4}(H_{12}(J)J^D J^D q^A - H_{12}(J)q^A J^D J^D - J^D J^D q^A H_{12}(J) + q^A J^D J^D H_{12}(J))|\kappa(j)\rangle \\
&= -\frac{1}{2}(j(H_{12}(j) - H_{12}(j-1))|\chi^A(j-1)\rangle + (j+1)(H_{12}(j+1) - H_{12}(j))|\rho^A(j+1)\rangle) \\
&= (|\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle) = q^A|\kappa(j)\rangle \,,
\end{aligned}
\tag{3.24}
$$

which requires

$$j\,(H_{12}(j) - H_{12}(j-1)) = -2\,. \tag{3.25}$$

We can solve (3.25) by adding it together for different values of integer j to find

$$\tfrac{1}{2}(H_{12}(j) - H_{12}(0)) = -\sum_{n=1}^{j}\frac{1}{n}\,, \tag{3.26}$$

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whose solution is proportional to the digamma function up to an additive constant,

$$H_{12}(J) = -2(\psi(J+1) - \psi(1)) \,. \tag{3.27}$$

The overall minus sign in (3.27) results from the normalization (3.6), and corresponds to the ferromagnetic case with $\beta = 2$. Following [40], it is often useful to think of $\psi(J+1)$ as a polynomial in $J_1^A J_2^A$, given by $f(J_1^A J_2^A)$, which is equal to $\psi(J+1)$ at the eigenvalues of $J_1^A J_2^A$. Using the value of $\psi(k+1) \equiv \sum_{n=1}^{k} \frac{1}{n} - \gamma$ for certain values of k (the non-negative integers), one can write a Lagrange interpolating polynomial for $\psi(J+1)$ in terms of the nearest neighbor pair $J_1^A J_2^A$, which has eigenvalues $x_j = s(s+1) - \frac{1}{2}j(j+1)$,

$$\psi(J+1) = \sum_{k=0}^{2s} c_k (J_1^A J_2^A)^k = f_{2s}(J_1^A J_2^A) \,, \tag{3.28}$$

and where the polynomial $f_{2s}(x)$ is

$$f_{2s}(x) = \sum_{k=0}^{2s} \left(\psi(k+1) \prod_{j=0,j\neq k}^{2s} \frac{x - x_j}{x_k - x_j} \right). \tag{3.29}$$

In particular, for $s = \frac{1}{2}$, $f_{2s}(x) = -x + \frac{3}{4} - \gamma$ so we regain $H_{12} = 2J_1^A J_2^A$, up to a constant.

When $s \neq 0, \frac{1}{2}, 1, \ldots$, the spin model can have local spin variables J_i^A taking values in an infinite-dimensional space: J_i^A acts on V_S where V_S is an infinite-dimensional spin s-representation and $V_S \otimes V_S = \sum_{j=0}^{\infty} V_j$, and V_j are the two-site infinite-dimensional irreducible representations. The Hamiltonian (3.27) and R-matrix (3.20) are continued to general s, for any real form of $SU(2)$. As discussed in [5,2], a non-compact $SL(2)$ subsector of the four-dimensional planar superconformal Yang-Mills theory leads to a one-loop anomalous dimension operator H_{12} whose eigenvalues can be identified with $\sum_{j=0}^{\infty}(2\psi(j+1)+2\gamma)$. The action of the local spin variable J_i^A on the one-particle states corresponds to $s = -\frac{1}{2}$, and one concludes that in this subsector of states, the one-loop anomalous dimension operator is the $XXX_{-\frac{1}{2}}$ Hamiltonian for the Lie algebra $SU(1,1)$. In this case, (3.6) becomes $J^A J^A = \frac{1}{2}I_1 \otimes I_2 + 2J_1^A J_2^A = -J(J+1)$, and the spin of V_j is $-1 - j$.

4. $PSU(2,2|4)$ Integrable Spin Chain

Now we consider a spin model with local spin variables J_i^A satisfying the $PSU(2,2|4)$ superalgebra with structure constants f_{ABC},

$$[J_i^A, J_j^B\} = f_C^{AB} J_j^C \delta_{ij} \,, \tag{4.1}$$

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in the representation where J_i^A acts on the space V_F. V_F is spanned by one-particle states in free $D = 4$, $N = 4$ superconformal Yang-Mills theory radially quantized on $\mathbf{R} \times S^3$, and is an infinite-dimensional representation of $PSU(2,2|4)$ with $J_1^A J_1^A V_F = 0$.

The two-particle Casimir, with $J^A = J_1^A + J_2^A$, is given by

$$J^A J^A = J(J + 1) \,. \tag{4.2}$$

The tensor product decomposition is

$$V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j \,, \tag{4.3}$$

so the eigenvalue of J takes values $j = 0, 1, 2, \ldots, \infty$. V_j are $PSU(2,2|4)$ irreducible infinite-dimensional representations corresponding to the two-particle states. Note that (4.2) holds in the two-particle sector of $PSU(2,2|4)$ SYM theory, as though the algebra were $SU(2)$. We have chosen a positive normalization in (4.2), which is consistent with (3.6), since the the $PSU(2,2|4)$ Casimir contains the negative of the generators $R_b^a R_a^b$ of the $SU(2)$ Casimir [2,25].

Following the discussion in the previous section, we can derive the form of the one-loop $PSU(2,2|4)$ Hamiltonian from the tree level Yangian generator

$$Q^A = f_{BC}^A \sum_{i<j} J_i^B J_i^C \,, \tag{4.4}$$

by again requiring that the density $H_{12}(J)$ satisfies

$$[H_{12}(J), Q_{12}^A] = q_{12}^A \,, \tag{4.5}$$

for $q_{12}^A = J_1^A - J_2^A$. Using (3.13), (3.15) and (4.2), we find

$$\begin{aligned}
&[H_{12}(J), Q^A]|\kappa(j)\rangle \\
&= \frac{1}{4}(H_{12}(J)J^D J^D q^A - H_{12}(J)q^A J^D J^D - J^D J^D q^A H_{12}(J) + q^A J^D J^D H_{12}(J))|\kappa(j)\rangle \\
&= \frac{1}{2}(j(H_{12}(j) - H_{12}(j-1))|\chi^A(j-1)\rangle + (j+1)(H_{12}(j+1) - H_{12}(j))|\rho^A(j+1)\rangle) \\
&= (|\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle) = q^A|\kappa(j)\rangle \,,
\end{aligned} \tag{4.6}$$

where now the result (3.15) applies to the two-particle modules of the $PSU(2,2|4)$ gauge theory, as we already proved in [25]. Solving (4.6) requires

$$j\left(H_{12}(j) - H_{12}(j-1)\right) = 2 \,, \tag{4.7}$$

so adding (4.7) for different values of integer j, we find $\frac{1}{2}(H_{12}(j) - H_{12}(0)) = \sum_{n=1}^{j} \frac{1}{n}$, and

$$H_{12}(J) = 2(\psi(J+1) - \psi(1)).$$ (4.8)

This is the same Hamiltonian density derived from one-loop Feynman graphs in the super-conformal gauge theory [2,5]. For a chain of more than two spins, H is a sum of nearest neighbor terms $H_{i,i+1}$.

Now we extend the derivation of the R-matrix in the previous section to $PSU(2,2|4)$. Therefore we will introduce two appropriate representations, one for the auxiliary space and one for the quantum space. For our discussion, we will not need to distinguish between $PSU(4|4)$ and its real form, since we are using algebraic procedures. It will be useful to first consider the extended group $U(4|4)$. As in [26], we can define a single site representation given by the $4|4$ representation of the group $U(4|4)$, which has two extra $U(1)$ generators K and R. We will call this representation T_A. The $U(1)$ R-symmetry acts on the representation but is not a symmetry of the gauge theory, and the T_K generator acts by $T_K = 0$. We will also need the metric for $U(4|4)$ which we take to be $g_{AB} = \frac{1}{2}\mathrm{Str}\, T_A T_B$, where we now let $1 \leq A, B \leq 64$. We use the conventions for the supertraces in [26]. So $g_{KK} = g_{RR} = 0$, $g_{KR} \neq 0$; and when $A \neq K, R$ then $g_{AB} = 2\delta_{AB}$ and $g_{KA} = g_{RA} = 0$. This metric will be used to raise and lower the "A" index of the Lie algebra generators. The $4|4$ representation T_A has $[T_A, T_B] = f_{AB}^{C} T_C$, where f_{AB}^{C} are now the $U(4|4)$ structure constants. Then $f_{KB}^{A} = 0$ for all A, B since K is central and commutes with everything, and

$$f_{AB}^{R} = 0$$ (4.9)

for all A, B, since the $U(1)$ R-symmetry generator R never appears on the right hand side of the commutation relations. The metric is used to find $f_{A}^{RB} = 0 = f_{K}^{AB}$.

The second representation is defined by interpreting the infinite-dimensional module V_F as a representation of the extended group $U(4|4)$ as follows. We include the chirality operator B and the central charge C in the set J_i^{A} to form generators of $U(4|4)$. On the states in V_F, C acts as 0, and B gives the chiral charge.

We consider an abstract Yang-Baxter relation for a universal R-matrix $\mathcal{R}$, which is defined as an element in $\mathcal{A} \otimes \mathcal{A}$, when $\mathcal{A}$ is the Yangian algebra of $U(4|4)$. It is given by

$$\mathcal{R}_{12}\mathcal{R}_{32}\mathcal{R}_{31} = \mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}_{12}.$$ (4.10)

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We choose s' to denote $T_{A\,m}$ given in terms of the $4|4$ representation of $U(4|4)$, and s to denote that J_i^A acts on V_F. Then the analogue of (3.3) is

$$(I \otimes \rho(s,\lambda) \otimes \rho(s',\sigma))\mathcal{R}_{32} = R_{jm}(\sigma - \lambda) = (\sigma - \lambda + \frac{i}{2})I_j \otimes I_m + \eta J_j^A \otimes T_{A\,m}\,, \quad (4.11)$$

where η is an arbitrary normalization constant. Applying the representation $\rho(s,\lambda) \otimes \rho(s,\mu) \otimes \rho(s',\sigma)$ to (4.10) we write a linear equation for $R_{ij}(\lambda)$:

$$R_{ij}(\lambda - \mu)((\sigma - \mu)I_i \otimes I_m + \eta J_i^A \otimes T_{A\,m})((\sigma - \lambda)I_i \otimes I_m + \eta J_j^B \otimes T_{B\,m})$$
$$= ((\sigma - \lambda)I_i \otimes I_m + \eta J_j^B \otimes T_{B\,m})((\sigma - \mu)I_i \otimes I_m + \eta J_i^A \otimes T_{A\,m})R_{ij}(\lambda - \mu)\,. \tag{4.12}$$

As in the previous section, we set $\mu = 0$, and write i, j as $1, 2$. Then

$$R_{12}(\lambda)((\sigma J^A - \lambda J_1^A)T_{Am} + \eta J_1^A J_2^B (T_A T_B)_m)$$
$$= ((\sigma J^A - \lambda J_1^A)T_{Am} + \eta J_1^A J_2^B (T_B T_A)_m) R_{12}(\lambda)\,. \tag{4.13}$$

For a single site i, let $J_b^a \equiv J_i^A (T_A)_b^a$, where a, b label the matrix indices at the site m. We will use the property that the representation V_F satisfies the criterion at a given site i,

$$J_b^a J_c^b = \alpha J_c^a \tag{4.14}$$

modulo the identity δ_{ac}, for some proportionality constant α [25,26]. This occurs for any representation M, when $M \otimes \bar{M}$ contains the adjoint only once, such as the n-dimensional representation of $SU(n)$. Note that this criterion (4.14) was needed in order for (4.4) to satisy the Yangian Serre relations [26]. Normalizing $[T_A, T_B\} = f_{AB}^C T_C$, with f_{AB}^C here given by the $U(4|4)$ structure constants, and using (4.14), we have

$$\begin{aligned} J_1^A J_2^B (T_A T_B)_b^a &= \tfrac{1}{2} J_1^A J_2^B [T_A, T_B\}_b^a + \tfrac{1}{2}(J_1^A J_2^B + J_2^A J_1^B)(T_A T_B)_b^a \\ &= \tfrac{1}{2}Q^A T_A + \tfrac{1}{2}J^A J^B T_A T_B - \tfrac{1}{2}(J_1^A J_1^B + J_2^A J_2^B)T_A T_B \\ &= \tfrac{1}{2}Q^A T_A + \tfrac{1}{2}J^A J^B T_A T_B - \tfrac{1}{2}\alpha J^A T_A\,, \end{aligned} \tag{4.15}$$

where here Q^A denotes the representation of the $U(4|4)$ Yangian on two sites

$$Q^C = f_{AB}^C J_1^A J_2^B\,. \tag{4.16}$$

So (4.13) reduces to

$$R_{12}(\lambda)((\sigma J^A - \lambda J_1^A)T_{A\,m} + \eta(\tfrac{1}{2}Q^A T_{A\,m} + \tfrac{1}{2}J^A J^B (T_A T_B)_m + \alpha J^A T_{A\,m}))$$
$$= ((\sigma J^A - \lambda J_1^A)T_{A\,m} + \eta(-\tfrac{1}{2}Q^A T_{A\,m} + \tfrac{1}{2}J^A J^B (T_A T_B)_m + \alpha J^A T_{A\,m})) R_{12}(\lambda)\,. \tag{4.17}$$

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Using the reasoning from the previous section, we assume the R-matrix depends on J_1^A, J_2^A only through the $U(4|4)$ two-particle Casimir $J^A J^A$, so that $[R_{12}(\lambda), J^A] = 0$ and (4.17) reduces to

$$R_{12}(\lambda)(\eta Q^A - \lambda q^A)T_A = -(\eta Q^A + \lambda q^A)T_A R_{12}(\lambda), \tag{4.18}$$

with

$$q^A = J_1^A - J_2^A. \tag{4.19}$$

We look for a solution that acts on each irreducible V_j module in (4.3),

$$R_{12}(\lambda) = r(\lambda, J)P_{12} \tag{4.20}$$

where P_{12} permutes the fields at sites 12 in the module V_j. J is the $PSU(4|4)$ Casimir defined via (4.2), since the difference between the $U(4|4)$ Casimir and the $PSU(2,2|4)$ Casimir is only BC, and the central charge C acts as 0 on V_F. We argue that (4.18) becomes

$$r(\lambda, J)(-\eta Q^A + \lambda q^A) = -(\eta Q^A + \lambda q^A)r(\lambda, J), \tag{4.21}$$

for A restricted to the $PSU(4|4)$ indices $1 \le A \le 62$, since $K = T_K = 0$ in this representation, and $Q^R = 0$ from (4.16) and (4.9). Also, $q^R = 0$, since J_i^R is C, which acts as zero.

Using (3.13), (3.15) for $PSU(4|4)$, and (4.2), we make the now familiar argument and let (4.21) act on a two-particle state $|\kappa(j)\rangle$ contained in the module V_j defined in (4.3),

$$r(\lambda, J)(-\frac{\eta}{4}J^D J^D q^A + \frac{\eta}{4}q^A J^D J^D + \lambda q^A)|\kappa(j)\rangle = -(\frac{\eta}{4}J^D J^D q^A - \frac{\eta}{4}q^A J^D J^D + \lambda q^A)r(\lambda, J)|\kappa(j)\rangle. \tag{4.22}$$

to find

$$r(\lambda, J)(-\frac{\eta}{4}J^D J^D q^A + \frac{\eta}{4}q^A J^D J^D + \lambda q^A)|\kappa(j)\rangle$$
$$= r(\lambda, j - 1)(\lambda + \frac{\eta}{2}j)|\chi^A(j - 1)\rangle + r(\lambda, j + 1)(\lambda - \frac{\eta}{2}(j + 1))|\rho^A(j + 1)\rangle, \tag{4.23}$$

and

$$(-\frac{\eta}{4}J^D J^D q^A + \frac{\eta}{4}q^A J^D J^D - \lambda q^A)r(\lambda, J)|\kappa(j)\rangle$$
$$= (-\lambda + \frac{\eta}{2}j)r(\lambda, j)|\chi^A(j - 1)\rangle + (-\lambda - \frac{\eta}{2}(j + 1))r(\lambda, j)|\rho^A(j + 1)\rangle. \tag{4.24}$$

Equating the coefficients of $|\chi^A(j - 1)\rangle$, or equivalently those of $|\rho^A(j + 1)\rangle$, we have

$$(\frac{\eta}{2}j + \lambda)r(\lambda, j - 1) = (\frac{\eta}{2}j - \lambda)r(\lambda, j), \tag{4.25}$$

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so the R-matrix that satisfies (4.12) is

$$R_{12}(\lambda) = \frac{\Gamma(J + 1 + \frac{2}{\eta}\lambda)}{\Gamma(J + 1 - \frac{2}{\eta}\lambda)} P_{12}.$$

(4.26)

Since the values of the permutation are $P_{12} = (-1)^j$ on each V_j [25], we find that the R-matrix acts on $V_F \otimes V_F$ as $\mathbf{R}_{12}(\lambda)$ given by

$$
\begin{aligned}
\mathbf{R}_{12}(\lambda) &= (-1)^J \frac{\Gamma(J + 1 + \frac{2}{\eta}\lambda)}{\Gamma(J + 1 - \frac{2}{\eta}\lambda)} \\
&= \sum_{j=0} (-1)^j \frac{\Gamma(j + 1 + \frac{2}{\eta}\lambda)}{\Gamma(j + 1 - \frac{2}{\eta}\lambda)} P_{12j},
\end{aligned}
$$

(4.27)

where P_{12j} projects the fields at positions $i, i + 1$ to the module V_j. Forming the transfer matrix,

$$F_f(\lambda) = \mathrm{tr}_f R_{Lf}(\lambda) R_{L-1f}(\lambda) \dots R_{1f}(\lambda)$$

(4.28)

we can find the Hamiltonian again,

$$H = \sum_{i=1}^{L} 2(\psi(J_{i,i+1} + 1) - \psi(1)) \sim \frac{d}{d\lambda} \ln F_f(\lambda)|_{\lambda=0}.$$

(4.29)

Up to multiplicative factors independent of J, the R-matrix (4.27) is the same as that found by extrapolating from a subsector of the gauge theory and assuming uniqueness [3]. This is seen easily from the reflection formulas $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$ and $\psi(1 - z) = \psi(z) + \pi \cot \pi z$, and the integer eigenvalues of J.

Since (4.10) is an abstract equation that holds in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, when $\mathcal{A}$ is the Yangian algebra of $U(4|4)$, we could permute the indices on (4.10) to find

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

(4.30)

which for the representation $\rho(s', \lambda) \otimes \rho(s', \mu) \otimes \rho(s, \sigma)$ results in

$$R_{mn}(\lambda - \mu)R_{im}(\lambda)R_{in}(\mu) = R_{in}(\mu)R_{im}(\lambda)R_{mn}(\lambda - \mu).$$

(4.31)

This is similar to obtaining (2.17) in the XXX model [40]. It would be interesting to solve (4.31) for $R_{mn}(\lambda)$ in $PSU(4|4)$, but we do not do that here.

5. Commuting Hamiltonians for the Periodic Chain as Casimirs of the $PSU(2,2|4)$ Yangian

In this section we show how the bilocal Yangian generator, although itself not defined for periodic boundary conditions, can be used to find the Casimirs for the periodic chain. We recall that the Hamiltonian [2]

$$H = \sum_{i=1}^{L} 2\left(\psi(J_{i,i+1} + 1) - \psi(1)\right),\tag{5.1}$$

is the first Casimir operator of the $PSU(2,2|4)$ Yangian [25], where $J_{i,i+1}$ is the two-site quadratic Casimir of the ordinary symmetry $PSU(2,2|4)$ given by

$$(J_i^A + J_{i+1}^A)(J_i^A + J_{i+1}^A) = J_{i,i+1}(J_{i,i+1} + 1).\tag{5.2}$$

We will demonstrate below that the second Yangian Casimir is

$$U = \sum_{i=1}^{L}(\psi(J_{i,i+1} + 1)\psi(J_{i+1,i+2} + 1) - \psi(J_{i+1,i+2} + 1)\psi(J_{i,i+1} + 1)).\tag{5.3}$$

We note that U of (5.3) reduces to H_2 in (2.22) for the $SU(2)_{\frac{1}{2}}$ subsector of the gauge theory. U acts on three adjacent sites at a time, and can be defined for a chain of three or more independent sites.

We have periodic boundary conditions $J_i^A = J_{i+L}^A$, with L independent sites, and as usual

$$J^A = \sum_i J_i^A,\tag{5.4}$$

where the index i runs over the number of independent sites. Since H and U are functions of the ordinary Casimir $J_{i,i+1}$, then

$$[H, J^A] = 0, \qquad [U, J^A] = 0.\tag{5.5}$$

We will now show that U is a Casimir of the Yangian. As a first step, we review why H is a Casimir of the Yangian. We start with an open chain $\bar{H} = \sum_{i=1}^{L} 2(\psi(J_{i\,i+1} + 1) - \psi(1))$, where we do not have periodic boundary conditions, and there are $L+1$ independent sites. The Yangian is well defined, with

$$Q^A = f_{BC}^A \sum_{i<j} J_i^B J_j^C,\tag{5.6}$$

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and the i, j indices run over the number of independent sites, as long as $i < j$. We compute the commutation relation of $\bar{H}$ with the Yangian as

$$[\bar{H}, J^A] = 0 \,, \qquad [\bar{H}, Q^A] = q^A_{1\,L+1} = (J^A_1 - J^A_{L+1}) \,, \tag{5.7}$$

i.e. $\bar{H}$ commutes with the Yangian up to the edge effect $q^A_{1\,L+1}$. To prove (5.7) in [25], we used, for $1 \leq i \leq L$, that

$$\begin{aligned}
[H_{i,i+1}, Q^A] &\equiv [2\psi(J_{i,i+1} + 1), Q^A] \\
&= [2\psi(J_{i,i+1} + 1), f^A_{BC} J^B_i J^C_{i+1}] = q^A_{i,i+1} \,,
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
[H_{i,i+1}, J^A] &\equiv [\psi(J_{i,i+1} + 1), J^A] \\
&= [\psi(J_{i,i+1} + 1), J^A_i + J^A_{i+1}] = 0 \,,
\end{aligned} \tag{5.9}$$

here the second lines in (5.8) and (5.9) are due to the absence of cross terms *c.f.*(2.13). As an aside, notice that inversely, given the Yangian charge (5.6), then (5.8) could be used to derive $\bar{H}$ as in (4.6)-(4.8).

Now we compute similar quantities for the second Casimir U. Let $\bar{U} = \sum_{i=1}^{L}(\psi(J_{i,i+1} + 1)\psi(J_{i+1,i+2} + 1) - \psi(J_{i+1,i+2} + 1)\psi(J_{i,i+1} + 1))$ be an open chain version of (5.3), where $\bar{U}$ is defined on $L + 2$ independent sites. Since $\bar{U}$ is just a product of the ψ's, we can show

$$\begin{aligned}
[\bar{U}, J^A] &= 0 \,, \\
[\bar{U}, Q^A] &= [\bar{U}, f^A_{BC} \sum_{1 \leq i \leq j \leq L+2} J^B_i J^C_j] \\
&= -[q^A_{23}, \psi(J_{12} + 1)] + [q^A_{L,L+1}, \psi(J_{L+1,L+2} + 1)] \\
&= -[J^A_2, \psi(J_{12} + 1)] - [J^A_{L+1}, \psi(J_{L+1,L+2} + 1)] \\
&= [J^A_1, \psi(J_{12} + 1)] - [J^A_{L+1}, \psi(J_{L+1,L+2} + 1)] \\
&= [(J^A_1 - J^A_{L+1}), \psi(J_{12} + 1)] + [(J^A_1 - J^A_{L+1}), \psi(J_{L+1,L+2} + 1)] \\
&= [q^A_{1\,L+1}, (\psi(J_{12} + 1) + \psi(J_{L+1,L+2} + 1))] \,,
\end{aligned} \tag{5.10}$$

where here the commutator is non-trivial for $L + 2$ sites of the Yangian. Note that (5.7) also holds for J^A and Q^A defined on $L + 1$ or more sites, *eg.*

$$[\bar{H}, Q^A] = [\bar{H}, f^A_{BC} \sum_{1 \leq i \leq j \leq L+1} J^B_i J^C_j] = [\bar{H}, f^A_{BC} \sum_{1 \leq i \leq j \leq L+2} J^B_i J^C_j] = q^A_{1\,L+1} \,. \tag{5.11}$$

This follows from

$$[\bar{H}, f_{BC}^A \sum_{i=1}^{L+1} J_i^B J_{L+2}^C] = 0 \tag{5.12}$$

since $\sum_{i=1}^{L+1} J_i^B$ is the $PSU(2,2|4)$ generator of the $L+1$ spin system, and so commutes with $\bar{H}$. To derive (5.10), we employ identities such as $[q_{12}^A, \psi(J_{23}+1)] - [q_{34}^A, \psi(J_{23}+1)] = -[(J_2^A+J_3^A), \psi(J_{23}+1)] = 0$, that follow from (4.1) and (5.2). From (5.10), the commutator of $\bar{U}$ with the Yangian is zero up to edge effects.

We now identify the periodic chain expressions H and U as Casimirs of the Yangian by considering the commutators (5.7) and (5.10). These commutators involve the open chain versions $\bar{H}$ and $\bar{U}$, where the Yangian makes sense. Clearly we cannot impose periodic boundary conditions (2.14) before performing the commutators, since Q^A would not be defined. But, if *after* performing the commutators in (5.7), (5.10), we let $J_1^A = J_{L+1}^A$, then the difference operator $q_{1\,L+1}^A$ vanishes, and the commutators are all zero. Then we find the periodic chain expressions H and U in (5.1) and (5.3), by simply imposing the periodic boundary conditions (2.14) on $\bar{H}$ and $\bar{U}$. As an example, for $L = 3$, one finds $\bar{H} = 2(\psi(J_{12}+1)-\psi(1)+\psi(J_{23}+1)-\psi(1)+\psi(J_{34}+1)-\psi(1))$. Setting $J_1^A = J_4^A$ in $\bar{H}$, we recover $H = 2(\psi(J_{12}+1)-\psi(1)+\psi(J_{23}+1)-\psi(1)+\psi(J_{31}+1)-\psi(1))$, which is (5.1) for $L = 3$. Similarly, $\bar{U} = [\psi(J_{12}), \psi(J_{23})] + [\psi(J_{23}), \psi(J_{34})] + [\psi(J_{34}), \psi(J_{45})]$, and imposing $J_1^A = J_4^A$, $J_2^A = J_5^A$, we regain the periodic chain expression $U = [\psi(J_{12}), \psi(J_{23})] + [\psi(J_{23}), \psi(J_{31})] + [\psi(J_{31}), \psi(J_{12})]$ which is (5.3) for $L = 3$. It is in this sense that H and U are Casimirs of the Yangian. *

These arguments are evidence of a commuting family of operators defined by (4.28), where $[F_f(\lambda), J^A] = 0$, and $[F_f(\lambda), F_f(\mu)] = 0$. Since H and U are both elements in the expansion of $F_f(\lambda)$, it follows as usual that

$$[H, U] = 0. \tag{5.13}$$

A direct check that $[H, U] = 0$ using the expressions in (5.1) and (5.3) involves the representations of $J_{i,i+1}$ and would be much harder.

Note that our "open chain" expressions do not commute, that is $[\bar{H}, \bar{U}] \neq 0$, and therefore should not be confused with commuting Hamiltonians for the open chain, which vanish when the number of adjacent sites is odd [45,46].

---

* In this paper, we are referring to a set of local operators as Casimirs of the Yangian.

As $L \to \infty$, there will be a infinite number of the local abelian Hamiltonians for the periodic chain $H, U, \ldots$, which are related to higher nested commutators of the $\psi(J_{i,i+1}+1)$. Each of these will be Casimirs of the Yangian. Since the Yangian $Y(G)$ has a basis $\mathcal{J}_n^A$ where $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, and $\mathcal{J}_n^A$ arises in commutators of the Q^A's, any Casimir of J^A and Q^A is also a Casimir of $Y(G)$.

6. Action of the Casimirs on the Chiral Primary States

Like the Hamiltonian H, the second Yangian Casimir U also annihilates the chiral primary states, as we will demonstrate in this section. The lowest components of the chiral primary representations are built only from the scalar fields ϕ^I, where $1 \le I \le 6$. For L independent sites, this is the symmetric traceless product of L **6**'s, where traceless representations are defined by those which give zero when any two indices are contracted, *eg.* [47]. So for $L = 2, 3$, we have

$$|\lambda_2\rangle = \phi^I \phi^J + \phi^J \phi^I - \frac{1}{3}\delta^{IJ}\phi^M \phi^M \,, \tag{6.1}$$

$$
\begin{aligned}
|\lambda_3\rangle = \ & \phi^I \phi^J \phi^K + \phi^I \phi^K \phi^J + \phi^J \phi^I \phi^K + \phi^J \phi^K \phi^I + \phi^K \phi^I \phi^J + \phi^K \phi^J \phi^I \\
& - \frac{1}{4}\delta^{IJ}(\phi^M \phi^M \phi^K + \phi^M \phi^K \phi^M + \phi^K \phi^M \phi^M) \\
& - \frac{1}{4}\delta^{IK}(\phi^M \phi^M \phi^J + \phi^M \phi^J \phi^M + \phi^J \phi^M \phi^M) \\
& - \frac{1}{4}\delta^{JK}(\phi^M \phi^M \phi^I + \phi^M \phi^I \phi^M + \phi^I \phi^M \phi^M) \,.
\end{aligned}
\tag{6.2}
$$

Using the notation of [2,3], we write the Hamiltonian as

$$
\begin{aligned}
H &= \sum_{i=1}^{L} 2(\psi(J_{i,i+1} + 1) - \psi(1)) \equiv \sum_{i=1}^{L} 2\,h(J_{i,i+1}) \\
&= \sum_{i=1}^{L}\sum_{j=0}^{\infty} 2\,h(j)P_{i,i+1,\,j}
\end{aligned}
\tag{6.3}
$$

where $P_{i,i+1,\,j}$ projects the fields at positions $i, i+1$ to the module V_j. These modules, for $j = 0, 1, 2, \ldots$ label the irreducible representations of $PSU(2, 2|4)$ which describe the two-particle system in free $N = 4$ super Yang-Mills theory. They appear in the tensor product $V_F \otimes V_F = \sum_{j=0}^{\infty} V_j$, as discussed in (4.3). The $h(j) = \sum_{n=1}^{j} \frac{1}{n}$ are harmonic numbers (with $h(0) \equiv 0$).

For $L = 2$, we know $(J_1^A + J_2^A)^2|\lambda_2\rangle = 0$, since $|\lambda_2\rangle$ is the lowest component of the module V_j with $j = 0$, *i.e.* $J_{12}|\lambda_2\rangle = 0$, where $(J_1^A + J_2^A)^2 = J_{12}(J_{12} + 1)$. Hence, as is well known,

$$H|\lambda_2\rangle = 2h(0)(P_{12,\,0} + P_{21,\,0})|\lambda_2\rangle = 4h(0)|\lambda_2\rangle = 0 \,. \tag{6.4}$$

In fact H annihilates all states in the module V_0, the superconformal chiral primary irreducible representation for two sites, since all such states are accessed from $|\lambda_2\rangle$ by generators of $PSU(2,2|4)$ which commute with H. Since U is a three-site operator, it does not act on $|\lambda_2\rangle$.

The $L = 2$ states which are lowest weights of the modules V_j, with $j > 0$, are superconformal primaries but not *chiral* superconformal primaries. The $j > 0$ states are not protected, and they will receive quantum corrections to their conformal dimension.

Now we consider chiral primaries with more than two sites $|\lambda_L\rangle$. Since the chiral primary state is symmetric and traceless in all indicies, it follows that it is symmetric and traceless in each pair. States on a pair of sites have definite j as described by(4.3) from [2,25], and since each pair is symmetric and traceless in the scalar fields then $j = 0$:

$$P_{i,i+1,0}|\lambda_L\rangle = |\lambda_L\rangle \,, \tag{6.5}$$

and projections to all other values of j vanish,

$$P_{i,i+1,j}|\lambda_L\rangle = 0 \,, \qquad j \neq 0 \,. \tag{6.6}$$

Then

$$\begin{aligned}
H|\lambda_L\rangle &= \sum_{i=1}^{L}\sum_{j=0}^{\infty} 2\,h(j)P_{i,i+1,\,j}\,|\lambda_L\rangle \\
&= \sum_{i=1}^{L} 2\,h(0)P_{i,i+1,\,0}\,|\lambda_L\rangle = 0 \,,
\end{aligned} \tag{6.7}$$

since $h(0) = 0$. For *eg.*, for the three-site chiral primary state $|\lambda_3\rangle$. we find that the one-loop anomalous dimension vanishes, since

$$H|\lambda_3\rangle = \sum_{i=1}^{L}\sum_{j=0}^{\infty} 2\,h(j)P_{i,i+1,\,j}\,|\lambda_3\rangle \,, \tag{6.8}$$

and

$$P_{i,i+1,j}|\lambda_3\rangle = \delta_{j,0}|\lambda_3\rangle \,, \tag{6.9}$$

for $1 \leq i \leq 3$. To see this explicitly, we can rewrite the chiral primary (6.2) and identify it with $P_{12,0}|\lambda_3\rangle$:

$$
\begin{aligned}
|\lambda_3\rangle = {} & (\phi^I \phi^J + \phi^J \phi^I - \frac{1}{3}\delta^{IJ}\phi^M \phi^M)\,\phi^K \\
& + (\phi^I \phi^K + \phi^K \phi^I - \frac{1}{3}\delta^{IK}\phi^M \phi^M)\,\phi^J \\
& + (\phi^J \phi^K + \phi^K \phi^J - \frac{1}{3}\delta^{JK}\phi^M \phi^M)\,\phi^I \\
& - \frac{1}{4}\delta^{IJ}(\phi^M \phi^K + \phi^K \phi^M - \frac{1}{3}\delta^{KM}\phi^N \phi^N)\,\phi^M \\
& - \frac{1}{4}\delta^{IK}(\phi^M \phi^J + \phi^J \phi^M - \frac{1}{3}\delta^{JM}\phi^N \phi^N)\,\phi^M \\
& - \frac{1}{4}\delta^{JK}(\phi^M \phi^I + \phi^I \phi^M - \frac{1}{3}\delta^{MI}\phi^N \phi^N)\,\phi^M \\
= {} & P_{12,0}\,|\lambda_3\rangle\,,
\end{aligned}
\tag{6.10}
$$

i.e. when we project the fields at positions $1, 2$ to the module V_0, we find that $P_{12,0}|\lambda_3\rangle = |\lambda_3\rangle$ which checks that $P_{12,j}|\lambda_3\rangle = 0$ for $j \neq 0$. Since $|\lambda_3\rangle$ is totally symmetric, it follows that $P_{23,0}|\lambda_3\rangle = |\lambda_3\rangle$, and $P_{31,0}|\lambda_3\rangle = |\lambda_3\rangle$. Then from (6.8),

$$
H|\lambda_3\rangle = 2h(0)(P_{12,0} + P_{23,0} + P_{31,0})|\lambda_3\rangle = 0\,.
\tag{6.11}
$$

The second Casimir U also annihilates $|\lambda_L\rangle$, since U is a sum of commutators that fan out to a sum of products of the Hamiltonian densities $2h(J_{i,i+1})$, each of which annihilates the chiral primary. Again we look at this explicitly for the lowest component of the three-site chiral primary $|\lambda_3\rangle$, where

$$
\begin{aligned}
U|\lambda_3\rangle = {} & \sum_{i=1}^{L} [h(J_{i,i+1}), h(J_{i+1,i+2})]\,|\lambda_3\rangle \\
= {} & \sum_{i=1}^{L}\sum_{j=0}^{\infty}\sum_{j'=0}^{\infty} h(j)h(j')[P_{i,i+1,j}, P_{i+1,i+2,j'}]\,|\lambda_3\rangle \\
= {} & \sum_{j=0}^{\infty}(\,h(j)h(0)(P_{12,j}P_{23,0} - P_{23,j}P_{12,0} \\
& \qquad + P_{23,j}P_{31,0} - P_{31,j}P_{23,0} \\
& \qquad + P_{31,j}P_{12,0} - P_{12,j}P_{31,0}\,)\,)\,|\lambda_3\rangle = 0\,.
\end{aligned}
\tag{6.12}
$$

The fact that H and U annihilate the lowest component states $|\lambda_L\rangle$, extends to all states in the superconformal chiral primary modules, since $[H, J^A] = [U, J^A] = 0$.

We conjecture that all the Casimirs of the Yangian annihilate the chiral primaries. This would explain why we see only the ordinary $PSU(2,2|4)$ symmetry in the supergravity Lagrangian of the AdS/CFT dual theory.

7. Higher-Loop Corrections

We have conjectured in [25] that the $PSU(2,2|4)$ Yangian generators can be defined to all orders in g^2N in the planar limit of the $\mathcal{N}=4$ SYM, where the exact generators obey

$$[\tilde{J}^A, \tilde{J}^B] = f_C^{AB}\tilde{J}^C, \qquad [\tilde{J}^A, \tilde{Q}^B] = f_C^{AB}\tilde{Q}^C, \tag{7.1}$$

together with the Serre relations [27,32]. Here we make a few comments on how (7.1) can be verified and used to probe higher-loop corrections.

The second equation in (7.1) expanded to one-loop is

$$[\delta J^A, Q^B] + [J^A, \delta Q^B] = f_C^{AB}\delta Q^C. \tag{7.2}$$

For $J^A = D$, (7.2) becomes

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B \delta Q^B, \tag{7.3}$$

where λ^B is the bare conformal dimension of J^B, and δD is the one-loop planar (spin chain) Hamiltonian that we have discussed before. The structure constants are given by the algebra, so they do not receive quantum corrections. Since $[D, \delta Q^B] = \lambda^B \delta Q^B$, and we checked in [25] that $[\delta D, Q^B] = 0$ modulo the edge effects which vanish for an infinite or periodic chain, this means that we have verified (7.1) through one-loop for all tree generators Q^A and a particular one-loop generator, the anomalous dimension operator $\delta D \equiv D_2 = H$.

We can also consider the one-loop correction to the non-local Yangian generators δQ^B. For the special bilocal Yangian generator associated with the dilatation index, $Q^B = Q^{(D)}$ in (7.2), such a correction must satisfy

$$[\delta J^A, Q^{(D)}] + [J^A, \delta Q^{(D)}] = -\lambda^A \delta Q^A. \tag{7.4}$$

We expect $\delta Q^{(D)}$ to depend on three sites at a time, and note that adding the second local Hamiltonian $\bar{U}$ to $\delta Q^{(D)}$ respects the algebraic constraints. But $\delta Q^{(D)}$ will also have non-local contributions. It could have a form similar to a non-local abelian Hamiltonian.

If we could compute the one-loop correction to the Yangian generator $\delta Q^A \equiv Q_2^A$, we could use it to check that

$$[D_4, Q^B] + [D_2, Q_2^B] = 0, \tag{7.5}$$

24

where D_2, D_4 are the one and two-loop corrections to the dilatation operator. This constraint follows from $[\tilde{J}^A, \tilde{Q}^B] = f_{ABC}\tilde{Q}^C$, which expanded to second order, $\mathcal{O}(g^2 N)^2$ in the 't Hooft coupling, and neglecting odd powers of g, is

$$[J_4^A, Q^B] + [J^A, Q_4^B] + [J_2^A, Q_2^B] = f_C^{AB} Q_4^C \,. \tag{7.6}$$

The ordinary symmetry analogue of (7.5) , $[D_4, J^B] + [D_2, J_2^B] = 0$, holds by inspection for the dilatation $J_2^{(D)} \equiv D_2$, since $[D, D_4] = 0$.

Next we comment on the possibility that the exact anomalous dilatation operator is an integrable Hamiltonian. Using the conjecture (7.1), we will show that the anomalous dimension ΔD to all orders in $g^2 N$ is a Casimir of the exact Yangian algebra. We consider the entire anomalous dilatation operator

$$\Delta D = \tilde{D} - D \,, \tag{7.7}$$

and note that since

$$[D, Q^B] = \lambda^B Q^B \,, \qquad [\tilde{D}, \tilde{Q}^B] = \lambda^B \tilde{Q}^B \,, \tag{7.8}$$

and higher-loop corrections retain their bare conformal dimension,

$$[D, \tilde{Q}^B] = \lambda^B \tilde{Q}^B \,, \tag{7.9}$$

we find that

$$[\Delta D, \tilde{Q}^B] = 0 \,. \tag{7.10}$$

This argument parallels one in [2] using the exact commutation relation for the ordinary symmetry generators, resulting in

$$[\Delta D, \tilde{J}^B] = 0 \,. \tag{7.11}$$

Thus we find from the Yangian defining relations (7.1), that a Casimir of the exact Yangian is given by ΔD. This strengthens the motivation to identify ΔD with a higher-loop integrable Hamiltonian Although the exact dilatation operator $\tilde{D}$ does not commute with the exact Yangian being one of its generators, the anomalous piece ΔD does. This suggests that the exact anomalous dilatation operator is an integrable Hamiltonian, which could be used to find the eigenvectors and eigenvalues of the states in the exact superconformal gauge theory.

Having probed the equations defining higher-loop corrections to the Yangian, we remark that if we were able construct the monodromy matrix with a suitable Yang-Baxter equation in terms of the exact Yangian generators $\tilde{J}^A, \tilde{Q}^A$, then the trace of the exact monodromy $\tilde{T}_m(u, g^2 N)$ could be reexpanded in u to find the family of Hamiltonians at higher loops, $\tilde{H}_\kappa(g^2 N)$. In some subsectors, commuting higher-loop higher Hamiltonians have been conjectured as a "best guess" to define higher-loop integrability [5,48], $[\tilde{H}_\kappa(g^2 N), \tilde{H}_\rho(g^2 N)] = 0$. This has been used to make an educated guess about the three-loop anomalous dimensions of certain operators. There is a possible discrepancy between this three-loop calculation and the corresponding three-loop term in the AdS/CFT dual string theory [49-59]. It would be interesting to find the connection between those higher-loop Hamiltonians and the higher-loop Yangian [60-63].

One might speculate that if we were to replace J^A and Q^A by the exact generators $\tilde{J}^A$, $\tilde{Q}^A$, then (2.24) would become the two-site monodromy matrix for the exact planar superconformal Yang-Mills theory, restricted to the $SU(2)$ subsector, with a similar extension to $L > 2$. But for that speculation to hold, $\tilde{J}^A$ and $\tilde{Q}^A$ would have to satisfy the condition (2.25) , as well as the Yangian defining relations. Possibly the extension of the monodromy to higher loops is more complicated than this, and involves additional $g^2 N$ dependence beyond the simple dependence through exact Yangian generators.

8. Conclusions and Outlook

Many of the features of integrable spin systems, such as a spin chain expression for the Hamiltonian, its commuting family of local Hamiltonians, the non-abelian Yangian generators (including the ordinary symmetry generators), the R-matrix $R_{ij}(u)$, the monodromy matrix $T_m(u)$, and the trace of the monodromy matrix $F(u) = \text{tr}_m T_m(u)$ can be constructed for the four-dimensional planar $\mathcal{N} = 4$ superconformal $SU(N)$ gauge theory. The techniques we use are novel in that they rely heavily on the Yangian. Gauge invariant states in planar Yang-Mills theory correspond to operators with a single trace. In their spin model description, this is reflected by periodic boundary conditions and a zero momentum condition. Although our representation for the tree level Yangian cannot be defined for periodic boundary conditions, we found that these techniques are still useful in the gauge theory.

The way the explicit *spin chain* expressions appear in the gauge theory is by the non-abelian non-local Yangian generators taking a tree level role, while the abelian local

Hamiltonians are identified with one-loop expressions. In conventional spin models, the analytic properties of the monodromy matrix link these two sets of symmetries together [31]. In the gauge theory, since the interpretation of the structures now involve both tree and one-loop quantities, this monodromy intertwines the two lowest orders of perturbation theory.

The two sets of symmetries, abelian and non-abelian, are also linked in the gauge theory by the identification of the periodic chain one-loop dilatation operator H, and the second local abelian Hamiltonian U, as Casimirs of the Yangian. The fact that H is a Casimir was seen to follow from consistency conditions which arose in an expansion to one-loop, when the $\mathcal{N} = 4$ Yang-Mills theory was assumed to have Yangian symmetry for all g^2N [25]. Together with the appearance of the non-abelian symmetry at $g^2N = \infty$, as indicated in the dual string theory[38,64-71], these results provide evidence for the presence of Yangian generators at higher loops. Their Casimir operators will then have higher-loop contributions, and they will form a commuting set of periodic chain higher-loop Hamiltonians. These will be local in the sense that the interactions involving a certain number of sites will vanish for sites far enough apart, since for periodic boundary conditions that feature leads to a vanishing commutator with the Yangian generators. One of the exact Casimirs is the exact anomalous dimension operator ΔD.

In conclusion, the Yangian is an important tool, and computing its higher-loop corrections may clarify the formulation of higher-loop integrability. Of particular interest is the extension to large g^2N of the Bethe ansatz equations for the eigenvalues and eigenvectors of ΔD. It would also be valuable to develop the link between integrability and twistor space structures in Yang-Mills theory [72-74].

Appendix A. Action of q_{12}^A on Two-Spin States in the XXX_s Spin Model

Two-spin states in the XXX_s model form modules V_j in the tensor product decomposition

$$V_S \otimes V_S = \bigoplus_{j=0}^{2s} V_j \tag{A.1}$$

when $s = 0, \frac{1}{2}, 1, \ldots$. We define $q_{12}^A = J_1^A - J_2^A$, and we will show that $q_{12}^A V_j$ is contained in $V_{j+1} \oplus V_{j-1}$, as found earlier for the $PSU(2,2|4)$ gauge theory [25]. As before, we prove this in two parts: (1) $q_{12}^A V_j$ occurs in the direct sum of V_k with $k - j$ odd, and (2) $q_{12}^A V_j$ occurs in the direct sum of V_k with $|j - k| \leq 1$. Part (1) follows from the $SU(2)$

tensor product decomposition. Let σ be the operator that exchanges the two V_S modules in (A.1). The irreducible representations V_j are either symmetric or antisymmetric, and for a given value of s, $\sigma V_j = (-1)^{j+2s}V_j$. Since $\sigma q_{12}^A = -q_{12}^A$; and the action of σ on $q_{12}^A V_j$ must match that of σ on V_k, since $q_{12}^A V_j \in \oplus V_k$, we have

$$\sigma\, q_{12}^A V_j = (-1)^{j+1+2s} q_{12}^A V_j\,,$$
$$\sigma V_k = (-1)^{k+2s} V_k\,,$$

(A.2)

which implies $(-1)^{j+1} = (-1)^k$ so $k - j$ must be odd. This proves Part (1). For Part (2), it will be sufficient to consider only the highest weight state $|\lambda(j)\rangle$ in each V_j, since any state in V_j is related to $|\lambda(j)\rangle$ by the raising operator J^+ of $SU(2)$, and the commmutator of q^A with the raising operator is a linear combination of the q^A. An arbitrary state in $SU(2)$ is labelled by $|j, m\rangle$, where m is the J^3 eigenvalue, and j is the quadratic Casimir eigenvalue. In this notation, the highest weight state is $|\lambda(j)\rangle = |j, j\rangle$, with $J^+|j, j\rangle = 0$. We will prove directly that

$$q^+|j, j\rangle \in V_{j+1}\,,$$
$$q^3|j, j\rangle \in V_j + V_{j+1}\,,$$
$$q^-|j, j\rangle \in V_{j-1} + V_j + V_{j+1}\,,$$

(A.3)

which will verify Part (2). We first consider $q^+|j, j\rangle$ in (A.3). Since $J^3 q^+|j, j\rangle = [J^3, q^+]|j, j\rangle + jq^+|j, j\rangle = (j+1)|j, j\rangle$, then $q^+|j, j\rangle \in \oplus_{k \geq j+1} V_k$, since modules with $k \leq j$ do not contain states with $m = j + 1$ eigenvalues. Also, $J^+ q^+|j, j\rangle = 0$, so $q^+|j, j\rangle$ is a highest weight state, thus $q^+|j, j\rangle \in V_{j+1}$. Similarly, $J^3 q^3|j, j\rangle = jq^3|j, j\rangle$, and $J^+ J^+ q^3|j, j\rangle = 0$, so $q^3|j, j\rangle \in V_j + V_{j+1}$. Lastly, $J^3 q^-|j, j\rangle = (j-1)|j, j\rangle$ and $J^+ J^+ J^+ q^-|j, j\rangle = 0$, so $q^-|j, j\rangle \in V_{j-1} + V_j + V_{j+1}$ which completes the proof of Part (2).

References

[1] J. A. Minahan and K. Zarembo, "The Bethe-Ansatz for N = 4 Super Yang-Mills," JHEP **0303**, 013 (2003) hep-th/0212208.

[2] N. Beisert, "The Complete One-loop Dilatation Operator of N = 4 super Yang-Mills Theory", Nucl. Phys. B **676**, 3 (2004), hep-th/0307015.

[3] N. Beisert and M. Staudacher, "The N = 4 SYM Integrable Super Spin Chain," Nucl. Phys. B **670**, 439 (2003), hep-th/0307042.

[4] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, "Stringing Spins and Spinning Strings," JHEP **0309**, 010 (2003), hep-th/0306139.

[5] N. Beisert, C. Kristjansen and M. Staudacher, "The Dilatation Operator of N = 4 Super Yang-Mills Theory," Nucl. Phys. **B664**, 131 (2003), hep-th/0303060.

[6] N. Beisert, "The Dilatation Operator of N = 4 Super Yang-Mills Theory and Integrability," hep-th/0407277.

[7] A. Belitsky, A. Gorsky and G. Korchemsky, "Gauge / String Duality for QCD Conformal Operators," Nucl. Phys. B **667**, 3 (2003), hep-th/0304028.

[8] A. Belitsky, V. Braun, A. Gorsky and G. Korchemsky, "Integrability in QCD and Beyond," hep-th/0407232.

[9] A. M. Polyakov, "Confinement and Liberation," hep-th/0407209.

[10] A. M. Polyakov, "Conformal Fixed Points of Unidentified Gauge Theories," Mod. Phys. Lett. A **19**, 1649 (2004), hep-th/0405106.

[11] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. **B72**, 461 (1974).

[12] L. N. Lipatov, "High-Energy Asymptotics of Multicolor QCD and Exactly Solvable Lattice Models," JETP Lett. **59**, 596 (1994), hep-th/9311037.

[13] G. Korchemsky and L. Faddeev, 'High-energy QCD as a Completely Integrable Model," Phys. Lett. B **342** 311 (1995), hep-th/9404173.

[14] D. Berenstein, J. Maldacena, and H. Nastase, "Strings in Flat Space and PP Waves from $N = 4$ Super Yang Mills," JHEP **204**, 013 (2002), hep-th/0202021.

[15] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, "Anomalous Dimensions of Wilson Operators in N = 4 SYM theory," Phys. Lett. B **557** 114 (2003), hep-ph/0301021.

[16] F. A. Dolan and H. Osborn, "Superconformal Symmetry, Correlation Functions and the Operator Product Expansion," Nucl. Phys. B **629**, 3 (2002), hep-th/0112251.

[17] A. V. Belitsky, "Fine Structure of Spectrum of Twist-Three Operators in QCD," Phys. Lett. B **453** 59 (1999), hep-ph/9902361; "Integrability and WKB Solution of Twist-Three Evolution Equations," Nucl. Phys. B **558** 259 (1999), hep-ph/9903512; 'Renormalization of Twist-Three Operators and Integrable Lattice Models," Nucl. Phys. B **574** 407 (2000), hep-ph/9907420.

[18] V. M. Braun, S. E. Derkachov and A. N. Manashov, "Integrability of Three-Particle Evolution Equations in QCD," Phys. Rev. Lett. **81** 2020 (1998), hep-ph/9805225.

[19] V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, "Baryon Distribution Amplitudes in QCD," Nucl. Phys. B **553** 355 (1999), hep-ph/9902375.

[20] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, "Evolution Equations for Quark Gluon Distributions in Multi-color QCD and Open Spin Chains," Nucl. Phys. B **566** 203 (2000), hep-ph/9909539.

[21] A. V. Belitsky, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, "Super-conformal Operators in N = 4 super-Yang-Mills Theory," Phys. Rev. D **70**, 045021 (2004), hep-th/0311104.

[22] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, "Classical / Quantum Integrability in AdS/CFT," JHEP **0405**, 024 (2004), hep-th/0402207.

[23] R. Roiban and A. Volovich, "Yang-Mills Correlation Functions from Integrable Spin Chains," hep-th/0407140.

[24] V. A. Kazakov and K. Zarembo, "Classical/Quantum Integrability in the Non-Compact Sector of AdS/CFT," hep-th/0410105.

[25] L. Dolan, C. Nappi, and E. Witten, "A Relation Between Approaches to Integrability in Superconformal Yang Mills Theory", JHEP**0310**, 017 (2003), hep-th/0308089.

[26] L. Dolan, C. Nappi, and E. Witten, "Yangian Symmetry in $D = 4$ Superconformal Yang-Mills Theory", QTS3 Conference Proceedings, University of Cincinnati, September 2003, hep-th/0401243.

[27] V. Drinfel'd, "Hopf Algebras and the Quantum Yang-Baxter Equation", Sov. Math. Dokl. **32** 254 (1985).

[28] V. Drinfel'd, "A New Realization of Yangians and Quantized Affine Algebras", Sov. Math. Dokl. **36** 212 (1988).

[29] V. Drinfel'd, "Quantum Groups", Proceedings of the International Congress of Math, Berkeley, California, 1986, pp 798-820.

[30] D. Bernard, "An Introduction to Yangian Symmetries," Int. J. Mod. Phys. **B7**, 3517 (1993) hep-th/9211133.

[31] D. Bernard and O. Babelon, "Dressing Symmetries", Commun. Math. Phys. **149**, 279 (1992), hep-th/9111036.

[32] D. Bernard, "Hidden Yangian in 2D Massive Current Algebras," Commun.Math.Phys. **137**, 191 (1991).

[33] F. Haldane, Z. Ha, J. Talstra, D. Bernard, and V. Pasquier, "Yangian Symmetry of Integrable Quantum Chains with Long-Range Interactions and a New Description of States in Conformal Field Theory", Phys. Rev. Lett. **69**, 2021 (1992).

[34] N. J. MacKay, "On the Classical Origins of Yangian Symmetry in Integrable Field Theory," Phys. Lett. B **281**, 90 (1992).

[35] N. J. MacKay, "Introduction to Yangian Symmetry in Integrable Field Theory", hep-th/0409183.

[36] G. Mandal, N. V. Suryanarayana, and S.R. Wadia, "Aspects of Semiclassical Strings in AdS(5)," Phys. Lett. **B543**, 81 (2002), hep-th/0206103.

[37] R. R. Metsaev and A. A. Tseytlin, "Type IIB Superstring Action in AdS(5) x S(5) Background," Nucl. Phys. **B533**, 109 (1998), hep-th/9805028.

[38] I. Bena, J. Polchinski and R. Roiban, "Hidden Symmetries of the AdS(5) x S^5 Superstring," hep-th/0305116.

[39] P. Kulish, N. Reshetikhin, E. Sklyanin, "Yang-Baxter Equation and Representation Theory 1", Lett. Math. Phys. **5** 393 (1981).

[40] L. Takhtajan and L. Faddeev, *Hamiltonian Methods in the Theory of Solitons*, Springer Series in Soviet Mathematics, Springer Verlag, 1987; L. Faddeev, "How Algebraic Bethe Ansatz works for Integrable Models", hep-th/9605187; L. Faddeev, "Algebraic Aspects of Bethe-Ansatz", hep-th/9404013.

[41] L. Faddeev, "Integrable Models in $(1 + 1)$-Dimensional Quantum Field Theory", in *Recent Advances in Field Theory and Statistical Mechanics*, Proceedings of the Les Houches Summer School 1982, North-Holland: Amsterdam 1984, pp. 561-608.

[42] M. Luscher, "Dynamical Charges in the Quantized, Renormalized Massive Thirring Model", Nucl. Phys. **B117**, 475 (1976).

[43] V. Tarasov, L. Takhtajan, and L. Faddeev, "Local Hamiltonians for Integrable Quantum Models on a Lattice", Theor. Math. Phys. **57**, 1059 (1983).

[44] M. Grabowski and P. Mathieu, "'Structure of the Conservation Laws in Integrable Spin Chains with Short Range Interactions," Annals Phys. **243**, 299 (1995) hep-th/9411045.

[45] E. Sklyanin, "Boundary Conditions for Integrable Quantum Systems", J. Phys. A: Math Gen. **21** (1988) 2375.

[46] M. Grabowski and P. Mathieu, "The Structure of Conserved Charges in Open Spin Chains", J. Phys. A: Math Gen. **29** (1996) 7635.

[47] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity," Phys. Rept. **323**, 183 (2000) hep-th/9905111.

[48] N. Beisert, "The $SU(2|3)$ Dynnamic Spin Chain", hep-th/0310252.

[49] D. Serban and M. Staudacher, "Planar $N = 4$ Gauge Theory and the Inozemtsev Long Range Spin Chain", JHEP **0406**, hep-th/0401067.

[50] G. Arutyunov and M. Staudacher, "Two-loop Commuting Charges and the String / Gauge Duality," hep-th/0403077.

[51] C. Callan, T. McLoughlin and I. Swanson, "Holography Beyond the Penrose Limit," Nucl. Phys. B **694**, 115 (2004) hep-th/0404007.

[52] C. G. . Callan, T. McLoughlin and I. Swanson, "Higher Impurity AdS/CFT Correspondence in the Near-BMN Limit," hep-th/0405153.

[53] I. Swanson, "On the Integrability of String Theory in AdS(5) x S**5,"hep-th/0405172.

[54] N. Mann and J. Polchinski, "Finite Density States in Integrable Conformal Field Theories," hep-th/0408162.

[55] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, "Precision Spectroscopy of AdS/CFT," JHEP **0310**, 037 (2003), hep-th/0308117.

[56] A. A. Tseytlin, "Semiclassical Strings and AdS/CFT," hep-th/0409296.

[57] D. Sadri and M. M. Sheikh-Jabbari, "The Plane-Wave / Super Yang-Mills Duality," hep-th/0310119.

[58] J. Engquist, J. A. Minahan and K. Zarembo, "Yang-Mills Duals for Semiclassical Strings on AdS(5) x S**5," JHEP **0311**, 063 (2003), hep-th/0310188.

[59] M. Lubcke and K. Zarembo, "Finite-size Corrections to Anomalous Dimensions in N = 4 SYM theory," JHEP **0405**, 049 (2004), hep-th/0405055.

[60] G. Arutyunov and M. Staudacher, "Matching Higher Conserved Charges for Strings and Spins," JHEP **0403**, 004 (2004), hep-th/0310182.

[61] J. Engquist, "Higher Conserved Charges and Integrability for Spinning Strings in AdS(5) x S**5," JHEP **0404**, 002 (2004), hep-th/0402092.

[62] M. Smedback, "Pulsating Strings on AdS(5) x S**5," JHEP **0407**, 004 (2004), hep-th/0405102.

[63] N. Berkovits, "BRST Cohomology and Nonlocal Conserved Charges," hep-th/0409159.

[64] A. M. Polyakov, "Interaction of Goldstone Particles in Two Dimensions. Applications to Ferromagnets and Massive Yang-Mills Fields," Phys. Lett. **B 59**, 79 (1975); "Hidden Symmetry Of The Two-Dimensional Chiral Fields," Phys. Lett **B72**, 224 (1977); "String Representations and Hidden Symmetries for Gauge Fields," Phys. Lett B82 (247) 1979; "Gauge Fields as Rings of Glue," Nucl. Phys. **B164**, 1971 (1980).

[65] M. Luscher and K. Pohlmeyer, "Scattering of Massless Lumps and Nonlocal Charges in Two-dimensional Classical Non-linear Sigma Model", Nucl. Phys. **B137**, 46 (1978).

[66] M. Luscher, "Quantum Non-local Charges and Absence of Particle Production in the Two-dimensional Non-linear σ Model", Nucl. Phys. **B135**, 1 (1978).

[67] L. Dolan, "Kac-Moody Algebra is Hidden Symmetry of Chiral Models," Phys. Rev. Lett. **47** 1371 (1981); "Kac-Moody Symmetry Group of Real Self-dual Yang-Mills," Phys. Lett. **113B**, 378 (1982); "Kac-Moody Algebras and Exact Solvability in Hadronic Physics", Phys. Rep. **109**, 1 (1984).

[68] P. . Goddard and D. I. Olive, "Kac-Moody And Virasoro Algebras: A Reprint Volume For Physicists," Adv. Ser. Math. Phys. **3** (1988) 1.

[69] J. H. Schwarz, "Classical Symmetries of Some Two-dimensional Models Coupled to Gravity," Nucl. Phys. B **454**, 427 (1995), hep-th/9506076.

[70] B. C. Vallilo, "Flat Currents in the Classical AdS$_5 \times S^5$ Pure Spinor Superstring," hep-th/0307018.

[71] N. Berkovits, "Super-Poincare Covariant Quantization of the Superstring," JHEP **0004**, 018 (2000), hep-th/0001035.

[72] E. Witten, "Perturbative Gauge Theory as a String Theory in Twistor Space," hep-th/0312171.

[73] F. Cachazo, P. Svrcek and E. Witten, "MHV Vertices and Tree Amplitudes in Gauge Theory," JHEP **0409**, 006 (2004), hep-th/0403047.

[74] L. J. Mason and N. M. J. Woodhouse, *Integrability, Self-Duality, and Twistor Theory*, Oxford, U.K. Clarendon (1996) 364 p. London Mathematical Society monographs, new series: 15.